Ergodic capacity, outage capacity, and information transmission over Rayleigh fading channels

Sayantan Choudhury and Jerry D. Gibson
Department of Electrical and Computer Engineering
University of California, Santa Barbara
Email: {sayantan, gibson}@ece.ucsb.edu

***Abstract*— We consider the lossy transmission of source information over Rayleigh fading channels. We investigate the usefulness of various channel capacity definitions, namely ergodic capacity, where it is assumed that the channel transitions over all the fading states, and outage capacity, where the source is transmitted at a constant rate with a specified outage probability. We also study the outage rate and expected source distortion for different outage probabilities. It is observed that the outage probabilities required to maximize outage rate and minimize expected distortion are quite different. This implies that schemes based on maximizing capacity might not lead to the most efficient design for the lossy transmission of source information over wireless networks. We also show that minimizing expected distortion over a wireless link does not necessarily minimize the variance of the distortion, and hence parameter selection based on minimizing expected distortion can lead to a high distortion for a specific realization. Finally, we observe that in a Rayleigh fading channel, in addition to a capacity distribution, there is a source distortion distribution at the receiver for a memoryless Gaussian source. A careful investigation of the capacity and source distortion distributions reveal that the probability of achieving the expected source distortion increases with an increase in average signal to noise ratio (SNR) unlike the case of ergodic capacity.**

I. Introduction

Recently, there has been considerable interest in source transmission over wireless networks. Several cross-layer design schemes have been proposed that improve the physical, link and network layers using a joint optimization framework [1]. There has also been some theoretical interest in evaluating source fidelity over a multihop channel [2], and in comparing source and channel diversity for various channel conditions [3].

In this paper, we compare the source distortion for two definitions of channel capacity, ergodic capacity and outage capacity, with and without channel state information (CSI) at the transmitter [4]. Ergodic capacity assumes that the fading transitions through all possible fading states, and thus might not be very useful in practice for source transmission with fixed delay constraints. Outage capacity transmits at the maximum rate for a specified outage probability. Both of these definitions lead to different source distortion at the receiver. We also observe that the outage probability that maximizes outage rate is quite different from the outage probability that minimizes source distortion. This suggests that schemes maximizing ergodic capacity or outage rate might not necessarily be optimal for transmitting information source with high fidelity over wireless networks.

This research has been supported by the California Micro Program, Applied Signal Technology, Dolby Labs, Inc., Mindspeed, and Qualcomm, Inc., by NSF Grant Nos. CCF-0429884 and CNS-0435527, and by the UC Discovery Grant Program and Nokia, Inc.

In [2], the optimal transmission rate to minimize the expected received distortion is obtained for single and multihop wireless connections. In this paper, we show that the optimal rate to minimize expected distortion at the receiver can produce a large variance in the distortion. Hence in practice, schemes designed to minimize expected distortion can perform quite poorly for individual realizations. We also evaluate the distribution of achieved source distortion for an $i.i.d$ Gaussian source transmitted over a Rayleigh fading link with CSI available at both the transmitter and receiver. It is observed that the expected distortion cannot be guaranteed with high reliability at low SNRs.

The paper is outlined as follows. In the next section, we provide a brief description of the different notions of channel capacity used in wireless communications and evaluate the expected source distortion for different scenarios. In Section III, we compare the expected distortion at the receiver for two schemes of source transmission, namely, maximizing the outage rate and minimizing the expected distortion, respectively. We also compare the expected distortion and the variance of the distortion for a single-hop channel. In Section IV, we study the distribution of capacity and source distortion for different SNRs and observe that the expected distortion cannot be achieved with a high reliability for low SNRs. Section V states some conclusions.

II. Ergodic capacity, outage capacity and expected distortion

We discuss the classical definitions of channel capacity for fading channels and its applications to source transmission over fading channels. The system model consists of a single hop channel as shown in Fig. 1. We provide a brief review of the channel capacity definitions for the case that CSI is unavailable at the transmitter but available at the receiver. A detailed description of the capacity of flat fading channels can be obtained in [4].

A. System Model

As shown in Fig. 1, we consider a discrete time channel with stationary and ergodic time varying gain 'a' and additive white

Gaussian noise (AWGN) 'n'. A block fading channel gain is assumed that remains constant over a blocklength and changes for different block lengths based on a Rayleigh distribution. At the receiver the instantaneous signal to noise ratio (SNR) γ is then given by an exponential distribution:

$$p_\gamma(\gamma) = \frac{1}{\bar{\gamma}} \exp(-\frac{\gamma}{\bar{\gamma}}), \qquad \gamma \geq 0 \tag{1}$$

where $\bar{\gamma}$ is the average SNR.

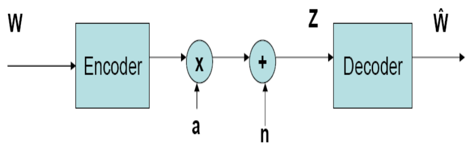


Fig. 1. System model

B. Channel Side information at Receiver

We consider the case where the CSI is known at the receiver, i.e., γ is known at the receiver for every time instant. In practice, this is accomplished using channel estimation techniques [5]. Moreover, the distribution of γ is known at both the transmitter at receiver. Traditionally, for capacity analyses, two channel capacity definitions are used, namely *ergodic capacity* and *outage capacity*.

1) Shannon (Ergodic) capacity: In this scenario, where the CSI is not available at the transmitter, the source data is transmitted at a constant rate. Since no CSI is available at the transmitter, data transmission takes place over all fading states including deep fades where the data is lost and hence the effective capacity is significantly reduced. The Shannon capacity of a fading channel with receiver CSI for an average power constraint $\bar{P}$ is given by [4]

$$C_{erg} = \int_0^\infty B \log_2(1+\gamma)p(\gamma)d\gamma \tag{2}$$

where B is the received signal bandwidth. This is also referred to as ergodic capacity since it is the average of the instantaneous capacity for an AWGN channel with SNR γ given by $B\log_2(1+\gamma)$.

For an $i.i.d$ Gaussian source sequence with mean zero and variance σ^2, the rate distortion function with squared error distortion is given by [6]

$$R(D) = \begin{cases} 0.5\log_2(\frac{\sigma^2}{D}), & 0 \leq D \leq \sigma^2 \\ 0, & D > \sigma^2 \end{cases} \tag{3}$$

Rewriting Eq. (3) , we can express the distortion in terms of the transmitted rate as

$$D(R) = \sigma^2 2^{-2R} \tag{4}$$

The distortion at the receiver for the constant rate transmission scheme is then given by

$$D(C_{erg}) = \sigma^2 2^{-2C_{erg}} \tag{5}$$

However, this notion of ergodic capacity might not be a suitable performance metric for evaluating distortion of sources with delay constraints. As pointed out in [7], a very long Gaussian codebook is required for achievability of Shannon capacity, the length being dependent on the dynamics of the fading process. In particular, it must be long enough for the fading to reflect its ergodic nature, i.e. the symbol time T must be much larger than the coherence time T_{coh}, defined to be the time over which the channel is significantly correlated.

2) Outage capacity: Outage capacity is used for slowly varying channels where the instantaneous SNR γ is assumed to be constant for a large number of symbols. Unlike the ergodic capacity scenario, schemes designed to achieve outage capacity allow for channel errors. Hence, in deep fades these schemes allow the data to be lost and a higher data rate can be thereby maintained than schemes achieving Shannon capacity, where the data needs to be correctly received over all fading states [4].

Specifically, a design parameter P_{out} is selected that indicates the probability that the system can be in outage, i.e. the probability that the system cannot successfully decode the transmitted symbols. Corresponding to this outage probability, there is a minimum received SNR, γ_{min}, given by $P_{out} = p(\gamma < \gamma_{min})$. For received SNRs below γ_{min}, the received symbols cannot be successfully decoded with probability 1, and the system declares an outage. Since the instantaneous CSI is not known at the transmitter, this scheme transmits using a constant data rate $C_{out} = B\log_2(1+\gamma_{min})$ which is successfully decoded with probability $1 - P_{out}$. Hence the average outage rate R_{out} correctly received over many transmission bursts is given by

$$R_{out} = (1 - P_{out})B\log_2(1+\gamma_{min}) \tag{6}$$

The expected distortion at the receiver for an $i.i.d$ $N(0,\sigma^2)$ source for the above scheme is given by [2]

$$E[D] = D(B\log_2(1+\gamma_{min}))(1-P_{out}) + \sigma^2 P_{out} \tag{7}$$

This can be interpreted as follows: either the source data is correctly decoded, resulting in a received distortion $D(B\log_2(1+\gamma_{min}))$, or there is an outage in which case the received variance is the source variance σ^2.

III. Information transmission over outage channels

We evaluate the end to end distortion for the outage channel and observe that maximizing outage rate does not minimize the expected distortion. Moreover, we also observe that the variance of the distortion at the receiver can be quite high for both schemes of information transmission namely, maximizing outage rate and minimizing expected distortion.

A. *Maximizing outage rate and minimizing expected distortion*

We revisit the outage capacity scenario described in the previous section in the context of information transmission with high fidelity. A Rayleigh fading channel is considered where the instantaneous SNR γ is exponentially distributed with mean $\bar{\gamma}$. The bandwidth B and variance σ^2 are normalized to unity. P_{out} is given by [8]

$$P_{out} = p(\gamma < \gamma_{min}) = 1 - \exp(-\frac{\gamma_{min}}{\bar{\gamma}}) \tag{8}$$

Reversing Eq. (8), we obtain

$$\gamma_{min} = -\bar{\gamma} \log(1 - P_{out}) \tag{9}$$

P_{out} is a design parameter so the average rate correctly received (R_{out}) in Eq. (6) can be maximized as a function of P_{out}. Similarly, the expected distortion in Eq. (7) can be minimized as function of P_{out}. Surprisingly, it turns out that the outage probabilities in the two scenarios are quite different.

Figure 2 plots the outage rate as a function of outage probability for an average SNR of 10 dB. In Fig. 3, we plot the expected distortion as a function of outage probability for the same average SNR of 10 dB. From Fig. 3, we observe that the outage rate tends to zero for very low and high outage probabilities. As the outage probability is reduced, γ_{min} decreases and hence for successful decoding over all the fading states, only a very low constant rate can be transmitted, whereas as the outage probability is increased, higher rates can be transmitted, but this results in more errors. Therefore, at both extremes of outage probability, the outage rate tends to zero. Similar conclusions can be made for expected distortion. Hence, there are optimal outage probabilities, p_c and p_d, that maximize outage rate and minimize expected distortion, respectively. From Fig. 2 we observe that an outage probability of 0.37 maximizes outage rate whereas an outage probability of 0.17 minimizes expected distortion as evident from Fig. 3. Lower outage probabilities are required for minimizing source distortion since by employing higher rates with higher outages, there is a law of diminishing returns. Source distortion is an exponentially decaying function of data rate and the lower source distortion obtained by employing higher data rates is offset by the fact that most of the transmissions are not successfully received. Figure 4 plots the expected distortion for various outage probabilities at an average SNR of 25 dB. As the average SNR increases, the outage probability to minimize expected distortion is lower and the outage probability region to achieve minimum expected distortion becomes narrower. This has a nice intuitive appeal and suggests that as the channel improves, the transmitted data rate should be so selected that lesser outage is allowed so that most of the source data can be successfully decoded instead of operating at higher outage probabilities, where the outage capacity is higher but there is a higher probability of decoding error at the receiver.

B. *Expected distortion and variance of distortion*

We evaluate the standard deviation of source distortion for the outage probability scenario previously described. The

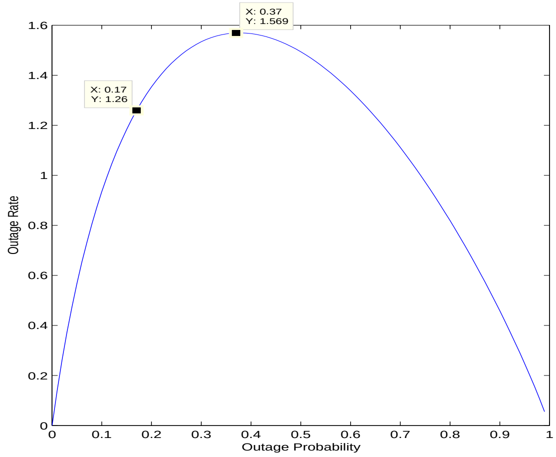


Fig. 2. Outage rate as a function of outage probability for an average SNR of 10 dB

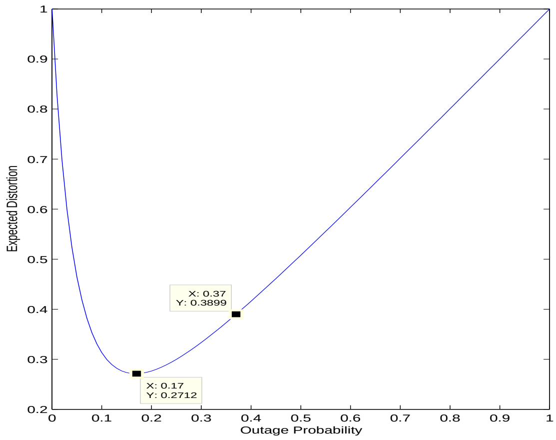


Fig. 3. Expected distortion as a function of outage probability for an average SNR of 10 dB

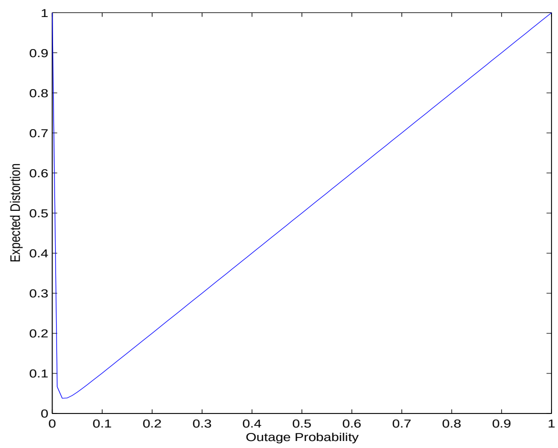


Fig. 4. Expected distortion as a function of outage probability for an average SNR of 25 dB

standard deviation of the source distortion at the receiver can be derived as

$$\sigma_{distn} = \sqrt{P_{out}(1 - P_{out})}\,(\sigma^2 - D(B\log_2(1+\gamma_{min}))) \quad (10)$$

In Fig. 5, we plot the standard deviation of the received distortion for an average SNR of 10 dB. We observe that the scheme that maximizes outage rate from Fig. 2 has a significantly higher standard deviation of distortion compared to the scheme that minimizes expected distortion from Fig. 3. However, even for the scheme that minimizes expected distortion, the variance of the distortion can be quite high. This implies that a particular channel realization might have a very high distortion though, on the average, the expected source distortion is minimized. In [9], we propose a performance indicator for speech transmission over wireless networks, MOS_x that guarantees a low distortion for a large percentage of realizations.

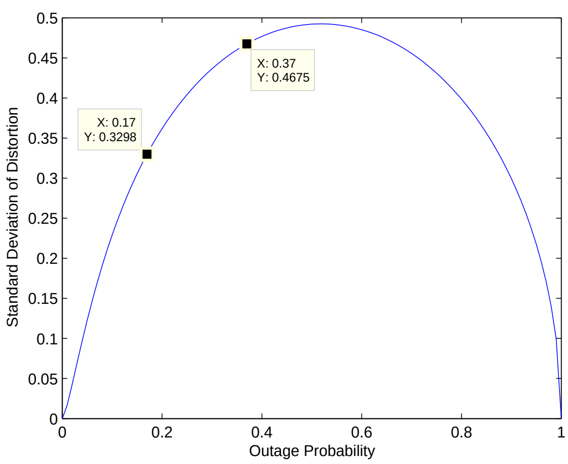


Fig. 5. Standard deviation of source distortion as a function of outage probability for an average SNR of 10 dB

IV. Source distortion with perfect CSI at both transmitter and receiver

In the previous section, we observed that schemes that maximize outage rate and minimize the expected distortion can have a high variance of source distortion at the receiver. In this section, we investigate the source distortion distribution at the receiver. For the scenario when the CSI is available only at the receiver and not at the transmitter, the source rate cannot be adapted based on channel conditions. Hence, at the receiver there are two cases, namely, the source is received correctly with a distortion $D(R_s)$, where R_s is the constant source transmission rate, or there is an outage with probability P_{out}, in which case the source distortion is σ^2. However, in the event that the CSI is available at the transmitter as well, the source rate can be adapted based on channel conditions, and this in turn induces a distribution of source distortion at the receiver as discussed below.

Let us consider the case where the CSI is available at the receiver as well as the transmitter. It is shown in [4] that the Shannon capacity in this scenario without power adaptation is the same as in Eq. (2), i.e. the transmitter side information does not increase the channel capacity unless the power is also adapted. However, since the CSI is available at the transmitter, the source rate can be adapted instantaneously and this leads to a much simpler and more practical system design. For instance, it is not required to have very long codes that average over all fading states. It is quite interesting to investigate the probability distribution of distortion induced at the receiver by this variable rate scheme and to determine, in particular, the probability of achieving the expected distortion.

The cumulative distribution function (CDF) of capacity can be derived as

$$F_c(c) = Pr\{C \leq c\} = 1 - \exp(\frac{1 - 2^{c/B}}{\bar{\gamma}}) \quad (11)$$

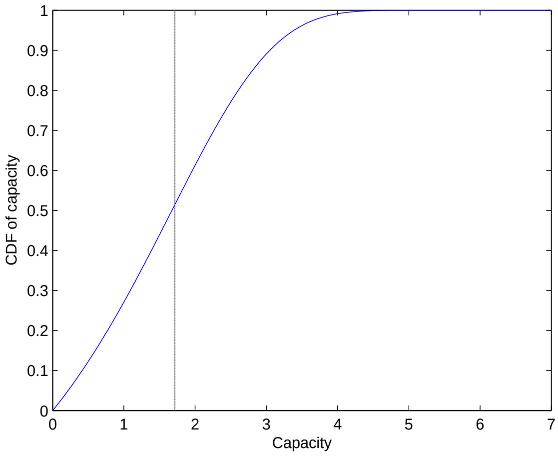
where B is the effective bandwidth and $\bar{\gamma}$ is the average SNR. The CDF for distortion can then be obtained as follows

$$\begin{aligned} F_d(d) &= Pr\{D \leq d\} = Pr\{\sigma^2 2^{-2C} \leq d\} \\ &= Pr\{-2C \leq \log_2 \frac{d}{\sigma^2}\} = Pr\{C \geq 0.5\log_2 \frac{\sigma^2}{d}\} \\ &= 1 - F_c(.5\log_2 \frac{\sigma^2}{d}) = \exp(\frac{1 - 2^{\frac{1}{2B}\log_2 \frac{\sigma^2}{d}}}{\bar{\gamma}}) \\ &= \exp(\frac{1 - (\frac{\sigma^2}{d})^{\frac{1}{2B}}}{\bar{\gamma}}) \end{aligned} \quad (12)$$

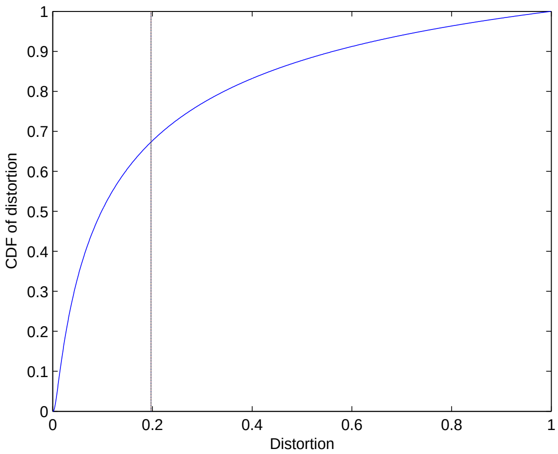
In Figs. 6 and 7, we plot the CDF for capacity and source distortion at the receiver for an average SNR of 5 dB and 15 dB, respectively. The vertical lines in the plots indicate the mean capacity and mean distortion, respectively. Comparing Figs. 6(a) and 7(a), we observe that for both the SNRs considered here, the mean capacity is achieved with a probability of nearly 0.5, and an increase of SNR increases the mean capacity value, but the probability of achieving it is quite similar in both cases. However, considering source distortion at the receiver, as shown in Figs. 6(b) and 7(b), an increase in SNR significantly improves its probability of exceeding the expected distortion. For an SNR of 5 dB, the expected distortion is achieved with a probability of 0.7 and an increase of SNR to 15 dB, not only reduces the overall expected distortion, but also increases the probability of achieving this value to nearly 0.9. The reason for the difference can be understood by investigating the probability density functions (pdfs). The pdf of capacity is roughly symmetric for both the SNRs considered whereas due to the exponentially decaying nature of the exponential function, the pdf of distortion is skewed toward the lower values of distortion.

V. Conclusions

We investigate the classical definitions of channel capacity with receiver CSI and no transmitter CSI for lossy transmission of source information over Rayleigh fading channels. We

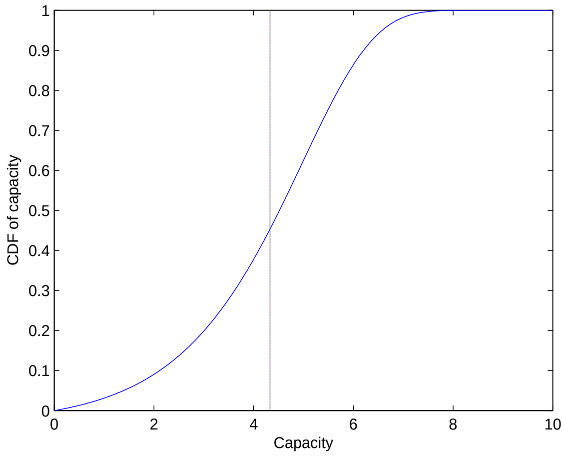


(a) Distribution of capacity

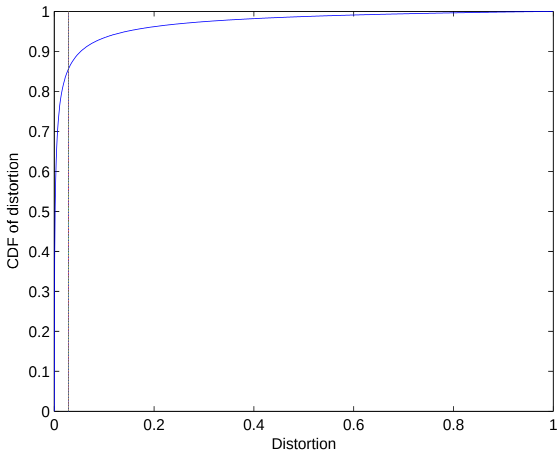


(b) Distribution of distortion

Fig. 6. CDF of capacity and distortion for an average SNR of 5 dB



(a) Distribution of capacity



(b) Distribution of distortion

Fig. 7. CDF of capacity and distortion for an average SNR of 15 dB

evaluate the source distortion at the receiver for two definitions of channel capacity, *ergodic capacity* and *outage capacity*, and observe that the outage probability required to maximize outage rate is quite different from minimizing source distortion at the receiver. We also observe that the scheme that maximizes outage rate has a larger standard deviation of source distortion at the receiver than the scheme that minimizes expected distortion. Finally, we evaluate the source distortion at the receiver for the case when CSI is available at the transmitter and the data rate is adapted to achieve instantaneous capacity. It is observed that the probability of achieving the mean distortion increases with an increase in SNR, which does not necessarily hold true for ergodic capacity.

References

[1] M. van der Schaar and S. Shankar, "Cross-layer wireless multimedia transmission: Challenges, principles, and new paradigms," *IEEE Wireless Communications Magazine*, vol. 12, no. 4, pp. 50–58, August 2005.

[2] K. E. Zachariadis, M. L. Honig, and A. K. Katsaggelos, "Source Fidelity over a Two-Hop Fading Channel," *Proc. MILCOM, International Conference on Military Communications*, pp. 134–139, November 2004.

[3] J. N. Laneman, E. Martinian, and G. W. Wornell, "Source-channel diversity approaches for multimedia communication," *ISIT 2004*, June-July 2004.

[4] A. Goldsmith, *Wireless Communication*. Cambridge University Press, 2005.

[5] J. Heiskala and J. Terry, *OFDM Wireless LANs: A Theoretical and Practical Guide* . Sams Publishing, December 2001.

[6] T. Berger, *Rate Distortion Theory*. Englewood Cliffs, NJ: Prentice- Hall, 1971.

[7] E. Biglieri, J. Proakis, and S. Shamai, "Fading channels: Information-theoretic and communication aspects," *IEEE Transactions on Information Theory*, vol. 44, no. 6, pp. 2619–2692, October 1998.

[8] A. Papoulis and S. U. Pillai, *Probability, Random Variables and Stochastic Processes*, 4th ed. McGraw-Hill Science, 2001.

[9] S. Choudhury, N. Shetty, and J. Gibson, "'MOSx and Voice Outage Rate in Wireless Communications," *40th Asilomar Conference on Signals, Systems and Computers*, Oct-Nov 2006.